

Today

- Binary addition
- Representing negative numbers

Binary Addition

Consider the following binary numbers:

0 0 1 0 0 1 1 0

0 0 1 0 1 0 1 1

How do we add these numbers?

Binary Addition

0 0 1 0 0 1 1 0

0 0 1 0 1 0 1 1



1

Binary Addition

0 0 1 0 0 1 1 0

0 0 1 0 1 0 1 1



0 1

And we have a carry now!

Binary Addition

0 0 1 0 0 1 1 0

0 0 1 0 1 0 1 1



0 0 1

And we have a carry again!

Binary Addition

0 0 1 0 0 1 1 0

0 0 1 0 1 0 1 1



0 0 0 1

and again!

Binary Addition

0 0 1 0 0 1 1 0

0 0 1 0 1 0 1 1



1 0 0 0 1

Binary Addition

0 0 1 0 0 1 1 0

0 0 1 0 1 0 1 1



0 1 0 0 0 1

One more carry!

Binary Addition

0 0 1 0 0 1 1 0

0 0 1 0 1 0 1 1



0 1 0 1 0 0 0 1

Binary Addition

Behaves just like addition in decimal, but:

- We carry to the next digit any time the sum of the digits is 2 (decimal) or greater

Negative Numbers

So far we have only talked about
representing non-negative integers

- What can we add to our binary representation that will allow this?

Representing Negative Numbers

One possibility:

- Add an extra bit that indicates the sign of the number
- We call this the “sign-magnitude” representation

Sign Magnitude Representation

+12 0 0 0 0 1 1 0 0

Sign Magnitude Representation

+12 0 0 0 0 1 1 0 0

-12 1 0 0 0 1 1 0 0

Sign Magnitude Representation

+12 0 0 0 0 1 1 0 0

-12 1 0 0 0 1 1 0 0

What is the problem with this approach?

Sign Magnitude Representation

What is the problem with this approach?

- Some of the arithmetic operators that we have already developed do not do the right thing

Sign Magnitude Representation

Operator problems:

- For example, we have already designed a counter (that implements an 'increment' operation)

-12

1 0 0 0 1 1 0 0

Sign Magnitude Representation

Operator problems:

-12

1 0 0 0 1 1 0 0



Increment

Sign Magnitude Representation

Operator problems:

-12

1 0 0 0 1 1 0 0



Increment

1 0 0 0 1 1 0 1

Sign Magnitude Representation

Operator problems:

-12

1 0 0 0 1 1 0 0

Increment

-13

1 0 0 0 1 1 0 1

!!!!

Representing Negative Numbers

An alternative:

(a little intuition first)

0

0 0 0 0 0 0 0 0



Decrement

Representing Negative Numbers

An alternative:

(a little intuition first)

0

0 0 0 0 0 0 0 0

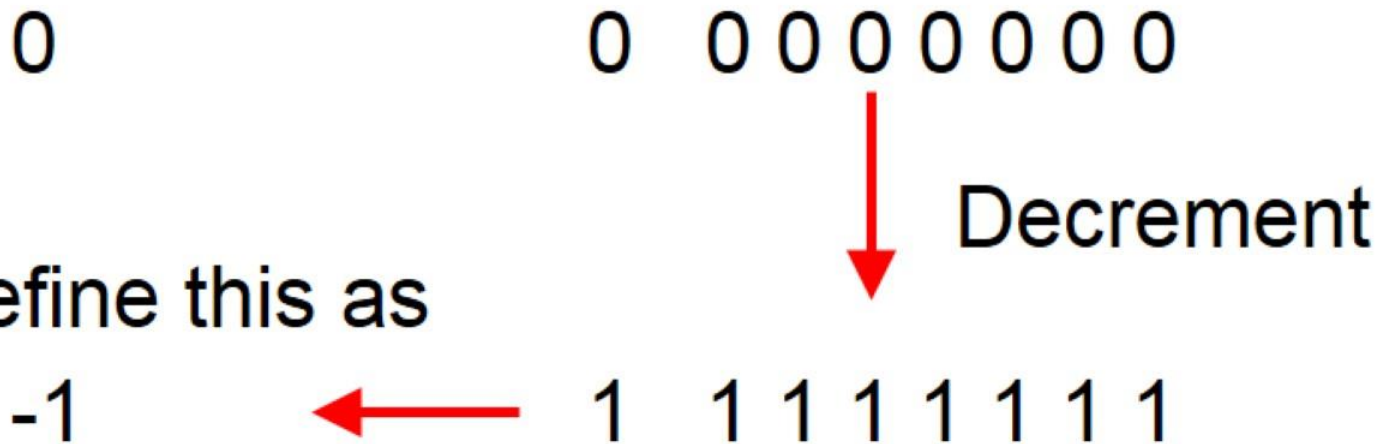


Decrement

1 1 1 1 1 1 1 1

Representing Negative Numbers

An alternative:
(a little intuition first)



Representing Negative Numbers

A few more numbers:

3	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0
1	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	0
-1	1	1	1	1	1	1	1	1
-2	1	1	1	1	1	1	1	0
-3	1	1	1	1	1	1	0	1

Two's Complement Representation

In general, how do we take the additive inverse of a binary number?

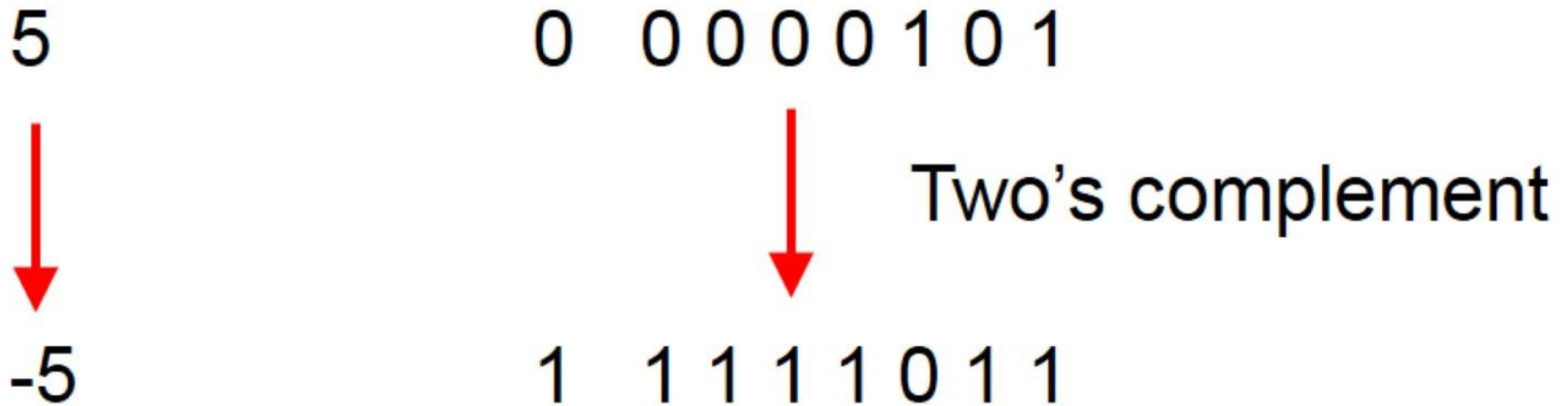
Two's Complement Representation

In general, how do we take the additive inverse of a binary number?

- Invert each bit and then add '1'

Two's Complement Representation

Invert each bit and then add '1'



Two's Complement Representation

Now: let's try adding a positive and a negative number:

$$\begin{array}{r} 12 \qquad \qquad 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \\ + \qquad \qquad \qquad + \\ -5 \qquad \qquad 1 \ 1 \ 1 \ 1 \ 1 \ 0 \ 1 \ 1 \end{array}$$

Two's Complement Representation

Now: let's try adding a positive and a negative number:

12	0	0	0	0	1	1	0	0
+					+			
-5	1	1	1	1	1	0	1	1
								
	0	0	0	0	0	1	1	1

Two's Complement Representation

Now: let's try adding a positive and a negative number:

12		0	0	0	0	1	1	0	0
+						+			
-5		1	1	1	1	1	0	1	1
									
7		0	0	0	0	0	1	1	1

Two's Complement Representation

Two's complement is used for integer representation in today's processors

Two's Complement Representation

Two's complement is used for integer representation in today's processors

One oddity: we can represent one more negative number than we can positive numbers

Implementing Subtraction

How do we implement a 'subtraction' operator?

(e.g., $A - B$)

Implementing Subtraction

How do we implement a 'subtraction' operator?

(e.g., $A - B$)

- Take the 2s complement of B
- Then add this number to A

Representing Fractions

Floating point representations are expensive:

- Require many bits
- Either require specialized hardware or long functions to compute mathematical operations

A Low-Cost Alternative: Fixed Point Representations

“w.f” fixed point:

- w bits to represent the whole number (including the sign)
- f bits to represent the fraction

A Low-Cost Alternative: Fixed Point Representations

“w.f” fixed point:

- We are representing values in units of 2^{-f}

So: 5.3 fixed point

- 5 bits for whole

A Low-Cost Alternative: Fixed Point Representations

5.3 fixed point (fits in an `int8_t`)

- 5 bits for whole
- 3 bits for fraction

What can we represent with this?

A Low-Cost Alternative: Fixed Point Representations

What can we represent with 5.3 fixed point?

- 5 bits for whole: 15 ... -16
- 3 bits for fraction: units of 1/8th

Fixed-Point Example

Fixed Point	Value	# of eighths
00000 000	0.0	0 eighths
00000 001		
00000 100		
00001 000		
00101 010		

Fixed-Point Example

Fixed Point		Value	# of eighths
000000	000	0.0	0 eighths
000000	001	0.125	1 eighth
000000	100	0.5	4 eighths
000001	000	1.0	8 eighths
00101	010	5.25	42 eighths

Adding Fixed-Point Numbers

```
int8_t a = 5;    // 5/8
```

```
int8_t b = 10;   // 10/8
```

```
int8_t c = a + b ???
```

$$5 \left(\frac{1}{8} s \right) + 10 \left(\frac{1}{8} s \right) = 15 \text{ what?}$$

Adding Fixed-Point Numbers

```
int8_t a = 5;      // 5/8  
int8_t b = 10;     // 10/8  
int8_t c = a + b;  // 15/8
```

$$5 \left(\frac{1}{8} s \right) + 10 \left(\frac{1}{8} s \right) = 15 \left(\frac{1}{8} s \right)$$

So: addition does the right thing

Multiplying Fixed-Point Numbers

```
int8_t a = 5;    // 5/8
```

```
int8_t b = 10;   // 10/8
```

```
int8_t c = a * b ???
```

$$5 \left(\frac{1}{8} s \right) \times 10 \left(\frac{1}{8} s \right) = 50 \text{ what?}$$

Multiplying Fixed-Point Numbers

```
int8_t a = 5;    // 5/8
```

```
int8_t b = 10;   // 10/8
```

```
int8_t c = a * b ???
```

$$5 \left(\frac{1}{8} s \right) \times 10 \left(\frac{1}{8} s \right) = 50 \left(\frac{1}{64} s \right)$$

But: we need to keep things in 5.3 format

Multiplying Fixed-Point Numbers

```
int8_t a = 5;      // 5/8  
int8_t b = 10;     // 10/8  
int8_t c = (a * b) >> 3; // 6/8
```

$$5 \left(\frac{1}{8} s \right) \times 10 \left(\frac{1}{8} s \right) = 50 \left(\frac{1}{64} s \right) \approx 6 \left(\frac{1}{8} s \right)$$

Dividing Fixed-Point Numbers

```
int8_t a = 20;    // 20/8  
int8_t b = 7;     // 7/8  
int8_t c = a / b ???
```

$$20 \left(\frac{1}{8} s \right) \div 7 \left(\frac{1}{8} s \right) = 2 \text{ What?}$$

Dividing Fixed-Point Numbers

```
int8_t a = 20;    // 20/8  
int8_t b = 7;     // 7/8  
int8_t c = a / b ???
```

$$20 \left(\frac{1}{8} s \right) \div 7 \left(\frac{1}{8} s \right) = 2 \text{ (1s)}$$

But: we want to stay within the 5.3 format. And – note that we have lost information in the rounding!

Dividing Fixed-Point Numbers

```
int8_t a = 20;    // 20/8  
int8_t b = 7;     // 7/8  
int8_t c = (a << 3) / b; // 160/7
```

$$20 \left(\frac{1}{8} s \right) \div 7 \left(\frac{1}{8} s \right) = 22 \left(\frac{1}{8} s \right)$$

Notes About the Book

The example code that the book gives tries to address some additional questions (but fails to be clear):

- In conversions from floating point to fixed-point, it catches errors when a floating point value is too small or too large to fit in the fixed point representation
- `assert(0)` just means that an error should be generated

Notes About the Book

- In the book, a “short” is 16 bits and a “long” is 32 bits.
- For many of the fixed-point examples, the fixed-point values fit in 16 bits
- After we perform a mathematical operation, it is possible that the result will not fit within the 16 bits
- So: all numbers are converted to 32 bits before the operation & the results are checked before converting back to 16 bits